

On Asymptotic Behaviour of Positive Solutions of $x'' = -t^{-\alpha/2-2}x^{1+\alpha}$ in the Superlinear Case

by

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Abstract. As a continuation work, we consider an initial value problem of the differential equation written in the title. For this, we transform the differential equation into a two dimensional autonomous system which has a centre as its critical point. We prove that this system has three kinds of orbits. Depending on these kinds, we get analytical expressions of the solution of the initial value problem which show its asymptotic behaviour. In the end, we state asymptotic behaviour of all positive solutions.

1. Introduction

Using the method of the papers [10], [11], we have been considering second order nonlinear differential equations

$$x'' = \pm t^{\alpha\lambda-2}x^{1+\alpha} \quad ('= d/dt) \quad (1.1)$$

where α, λ are real parameters. As stated in our previous papers [12] through [20], these differential equations are worth solving, for these can be applied to many fields – astrophysics, atomic physics, variational problems, partial differential equations, Riemannian geometry([5]), etc. Moreover many authors treated these (in the more general form) in [1], [3] through [9], for example, but they did not get all positive solutions.

In this paper, we take the sign “–” in the double sign of (1.1) and consider (1.1) in a region $0 < t < \infty, 0 < x < \infty$. For such (1.1), we have already treated the case $\alpha > 0$ in [12], [17] through [20] except the case $\lambda = -1/2$, and the case $\alpha < 0$ in [14], [15], [16]. That is, only the case $\alpha > 0, \lambda = -1/2$ remains and so we consider this case here. Then (1.1) has the form

$$x'' = -t^{-\alpha/2-2}x^{1+\alpha} \quad (E)$$

where $\alpha > 0$. Given an initial condition

$$\begin{aligned} x(t_0) = A, \quad x'(t_0) = B \\ (0 < t_0 < \infty, 0 < A < \infty, -\infty < B < \infty), \end{aligned} \quad (I)$$

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we shall show asymptotic behaviour of all solutions of the initial value problem (E), (I).

As done in our previous papers, we adopt a transformation

$$y = \psi(t)^{-\alpha} x^\alpha, \quad z = ty' \quad (\text{namely } x = \psi(t)y^{1/\alpha}) \quad (\text{T})$$

where $\psi(t) = 4^{-1/\alpha} t^{1/2}$ is a particular solution of (E), and reduce (E) into a first order rational differential equation

$$\frac{dz}{dy} = \frac{4(\alpha - 1)z^2 - \alpha^2 y^2(y - 1)}{4\alpha yz}. \quad (\text{R})$$

Moreover, using a parameter s we rewrite this as a two dimensional autonomous system

$$\frac{dy}{ds} = 4\alpha yz, \quad \frac{dz}{ds} = 4(\alpha - 1)z^2 - \alpha^2 y^2(y - 1) \quad (\text{S})$$

whose critical points are $(0, 0)$, $(1, 0)$ if $\alpha \neq 1$, and every point of the z axis, $(1, 0)$ if $\alpha = 1$. Since we consider positive solutions of (E), we always get $y > 0$ from (T).

Postponing the proof, we show the phase portrait of (S) in the figure below. In this figure, the bold curve denotes a unique orbit with properties

$$\lim_{y \rightarrow 0} \frac{z}{y} = \pm \frac{\alpha}{2}$$

which we call Γ , and the critical point $(1, 0)$ is a centre. Let Δ be the region Γ surrounds. Then Δ is filled with periodic orbits. The orbits lying outside $\Delta \cup \Gamma$ are continuable to $y = 0$ as $s \rightarrow \pm\infty$ and have properties

$$\lim_{y \rightarrow 0} \frac{z}{y} = \pm\infty.$$

Furthermore in the undrawn parts, if $0 < \alpha < 1$, $\alpha = 1$, $\alpha > 1$, then these orbits satisfy

$$\lim_{y \rightarrow 0} z = \pm\infty, \quad c, \quad 0$$

respectively, where c is a nonzero constant. We shall obtain analytical expressions of non-periodic orbits in the neighbourhood of $y = 0$ below.

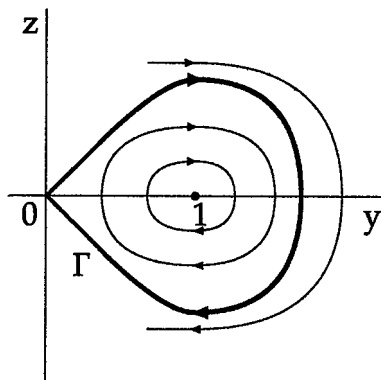


FIGURE. The phase portrait of (S)

Now, let $x(t)$ be a solution of (E), (I). Then if we apply (T) to $x(t)$, we get an orbit of (S) passing a point (y_0, z_0) where

$$y_0 = 4t_0^{-\alpha/2}A^\alpha, \quad z_0 = \alpha y_0 \left(-\frac{1}{2} + t_0 \frac{B}{A} \right).$$

So, noticing the position of (y_0, z_0) we state our theorems as follows:

THEOREM 1. *If $(y_0, z_0) \in \Delta$ and $(y_0, z_0) \neq (1, 0)$, then $x(t)$ is defined for $0 < t < \infty$ and determined as*

$$x(t) = 4^{-1/\alpha} t^{1/2} \left\{ 1 + \sum_{n=1}^{\infty} x_n(\theta) r_0^n \right\}. \quad (1.2)$$

Here $x_n(\theta)$ are 2π periodic functions of θ ,

$$r_0 = \sqrt{\frac{(y_0 - 1)^2}{8} + \frac{z_0^2}{2\alpha}},$$

and θ is an increasing function of t satisfying

$$\theta + Q(\theta, r_0) = K(r_0) \log t + C(t_0, \theta_0, r_0) \quad (1.3)$$

where $Q(\theta, r_0)$ is a continuous function of (θ, r_0) 2π periodic and differentiable in θ and holomorphic in r_0 , the mean of $Q(\theta, r_0)$ is zero as a function of θ , and $Q(\theta, 0) = 0$. Moreover $K(r_0)$ is a positive constant depending on r_0 holomorphically and

$$C(t_0, \theta_0, r_0) = \theta_0 + Q(\theta_0, r_0) - K(r_0) \log t_0,$$

$$\theta_0 = \tan^{-1} \frac{\sqrt{\alpha}(y_0 - 1) - 2z_0}{\sqrt{\alpha}(y_0 - 1) + 2z_0}.$$

The left hand side of (1.3) is an increasing function of θ .

From (1.3) we get

$$\theta \sim K(r_0) \log t$$

as $t \rightarrow 0, \infty$. Here $f(t) \sim g(t)$ means $\lim f(t)/g(t) = 1$. If we follow the discussion of Section 2, we can determine $K(r_0)$, $Q(\theta, r_0)$. Due to the increase property of the left hand side of (1.3), we might evaluate θ for given t . Moreover if $(y_0, z_0) = (1, 0)$, then we directly get $x(t) = \psi(t)$.

THEOREM 2. *If $(y_0, z_0) \in \Gamma$, then $x(t)$ is defined for $0 < t < \infty$. Furthermore $x(t)$ is represented as*

$$x(t) = Kt \left\{ 1 + \sum_{n=1}^{\infty} a_n t^{(\alpha/2)n} \right\} \quad (1.4)$$

in the neighbourhood of $t = 0$, and

$$x(t) = L \left\{ 1 + \sum_{n=1}^{\infty} b_n t^{-(\alpha/2)n} \right\} \quad (1.5)$$

in the neighbourhood of $t = \infty$. Here K, L, a_n, b_n are constants depending on t_0, A, B of (I).

THEOREM 3. If $(y_0, z_0) \notin \Delta \cup \Gamma$, then $x(t)$ is defined for $(0 <) \omega_- < t < \omega_+$ ($< \infty$). Moreover $x(t)$ is represented as

$$x(t) = K(t - \omega_-) \left\{ 1 + \sum_{m+n>0} a_{mn} (t - \omega_-)^m (t - \omega_-)^{an} \right\} \quad (1.6)$$

in the neighbourhood of $t = \omega_-$, and

$$x(t) = L(\omega_+ - t) \left\{ 1 + \sum_{m+n>0} b_{mn} (\omega_+ - t)^m (\omega_+ - t)^{an} \right\} \quad (1.7)$$

in the neighbourhood of $t = \omega_+$, where K, L, a_{mn}, b_{mn} are constants depending on t_0, A, B of (I).

These theorems will be proved in the following order: In Section 2, we shall show that the critical point $(1, 0)$ is a centre and a neighbourhood of this critical point are filled with periodic orbits. From these periodic orbits we shall get the analytical expression (1.2) of the solution of (E), (I). In Section 3, we shall obtain the analytical expressions (1.4), (1.5), (1.6), (1.7) from the other orbits of (S) and complete the proofs of Theorems 2, 3. Finally, in Section 4 we shall show that all orbits of Δ are those periodic ones, which will accomplish the proof of Theorem 1.

2. On the critical point $(1, 0)$

First we consider (S) in the neighbourhood of $(1, 0)$. For this, put

$$y = 1 + \eta, \quad z = \zeta.$$

Then we get

$$\begin{aligned} \frac{d\eta}{ds} &= 4\alpha\zeta + 4\alpha\eta\zeta, \\ \frac{d\zeta}{ds} &= -\alpha^2\eta - 2\alpha^2\eta^2 + 4(\alpha - 1)\zeta^2 - \alpha^2\eta^3 \end{aligned} \quad (2.1)$$

and $(1, 0)$ is transformed into $(0, 0)$. The characteristic exponents of (2.1) are $\pm 2\alpha^{3/2}i$. Hence the critical point $(0, 0)$ of (2.1) is a centre or a spiral point (see Theorem 4.1 of Chapter 13 of [2] for example). However if we rewrite (2.1) as

$$\frac{d\eta}{ds} = X(\eta, \zeta), \quad \frac{d\zeta}{ds} = Y(\eta, \zeta),$$

then we have

$$\frac{Y(\eta, -\zeta)}{X(\eta, -\zeta)} = -\frac{Y(\eta, \zeta)}{X(\eta, \zeta)},$$

which implies that orbits of (2.1) are symmetric with respect to the η axis. Therefore $(0, 0)$ is not a spiral point but a centre, namely a critical point $(1, 0)$ of (S) is a centre and a neighbourhood of $(1, 0)$ is filled with periodic orbits surrounding $(1, 0)$.

Now, put

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ \sqrt{\alpha} & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

in (2.1). Then we get

$$\begin{aligned} \frac{d}{ds} \begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} 0 & -2\alpha^{3/2} \\ 2\alpha^{3/2} & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &+ \begin{pmatrix} 2\alpha^{3/2}(u^2-v^2) - 4\alpha^{3/2}(u+v)^2 + 2(\alpha-1)\alpha^{1/2}(u-v)^2 - 4\alpha^{3/2}(u+v)^3 \\ 2\alpha^{3/2}(u^2-v^2) + 4\alpha^{3/2}(u+v)^2 - 2(\alpha-1)\alpha^{1/2}(u-v)^2 + 4\alpha^{3/2}(u+v)^3 \end{pmatrix}. \end{aligned}$$

Moreover, put

$$u = r \cos \theta, \quad v = r \sin \theta.$$

Then we have

$$\begin{aligned} \frac{d}{ds} \begin{pmatrix} r \\ \theta \end{pmatrix} &= \begin{pmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{pmatrix}^{-1} \\ &\left\{ \begin{pmatrix} 0 & -2\alpha^{3/2} \\ 2\alpha^{3/2} & 0 \end{pmatrix} \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} + \begin{pmatrix} a_2(\theta)r^2 + a_3(\theta)r^3 \\ b_2(\theta)r^2 + b_3(\theta)r^3 \end{pmatrix} \right\} \end{aligned}$$

where

$$\begin{aligned} a_2(\theta) &= -2\alpha^{1/2}(\alpha+1) + 2\alpha^{3/2} \cos 2\theta - 2(3\alpha-1)\alpha^{1/2} \sin 2\theta \\ a_3(\theta) &= -6\alpha^{3/2} \cos \theta - 6\alpha^{3/2} \sin \theta + 2\alpha^{3/2} \cos 3\theta - 2\alpha^{3/2} \sin 3\theta \\ b_2(\theta) &= 2\alpha^{1/2}(\alpha+1) + 2\alpha^{3/2} \cos 2\theta + 2(3\alpha-1)\alpha^{1/2} \sin 2\theta \\ b_3(\theta) &= -a_3(\theta). \end{aligned}$$

Furthermore, put

$$\begin{aligned} p_2(\theta) &= a_2(\theta) \cos \theta + b_2(\theta) \sin \theta, & p_3(\theta) &= a_3(\theta) \cos \theta + b_3(\theta) \sin \theta \\ q_1(\theta) &= -a_2(\theta) \sin \theta + b_2(\theta) \cos \theta, & q_2(\theta) &= -a_3(\theta) \sin \theta + b_3(\theta) \cos \theta. \end{aligned}$$

Then we obtain

$$\begin{aligned} p_2(\theta) &= (2\alpha-3)\alpha^{1/2} \cos \theta - (2\alpha-3)\alpha^{1/2} \sin \theta \\ &\quad - (2\alpha-1)\alpha^{1/2} \cos 3\theta - (2\alpha-1)\alpha^{1/2} \sin 3\theta \\ p_3(\theta) &= -4\alpha^{3/2} \cos 2\theta - 2\alpha^{3/2} \sin 4\theta \\ q_1(\theta) &= (6\alpha+1)\alpha^{1/2} \cos \theta + (6\alpha+1)\alpha^{1/2} \sin \theta \\ &\quad - (2\alpha-1)\alpha^{1/2} \cos 3\theta + (2\alpha-1)\alpha^{1/2} \sin 3\theta \\ q_2(\theta) &= 6\alpha^{3/2} + 8\alpha^{3/2} \sin 2\theta - 2\alpha^{3/2} \cos 4\theta \end{aligned}$$

and

$$\frac{d}{ds} \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 2\alpha^{3/2} \end{pmatrix} + \begin{pmatrix} p_2(\theta)r^2 + p_3(\theta)r^3 \\ q_1(\theta)r + q_2(\theta)r^2 \end{pmatrix}.$$

Here, put

$$\tau = 2\alpha^{3/2}s.$$

Then we get

$$\frac{d}{d\tau} \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} c_2(\theta)r^2 + c_3(\theta)r^3 \\ d_1(\theta)r + d_2(\theta)r^2 \end{pmatrix} \quad (2.2)$$

where

$$c_2(\theta) = \frac{p_2(\theta)}{2\alpha^{3/2}}, \quad c_3(\theta) = \frac{p_3(\theta)}{2\alpha^{3/2}}, \quad d_1(\theta) = \frac{q_1(\theta)}{2\alpha^{3/2}}, \quad d_2(\theta) = \frac{q_2(\theta)}{2\alpha^{3/2}}.$$

Hence if $1 + d_1(\theta)r + d_2(\theta)r^2 \neq 0$, then we get

$$\frac{dr}{d\theta} = \frac{c_2(\theta)r^2 + c_3(\theta)r^3}{1 + d_1(\theta)r + d_2(\theta)r^2}. \quad (2.3)$$

Since a critical point $(1, 0)$ is a centre, solutions of (2.3) are 2π periodic, if these exist in the neighbourhood of $r = 0$.

Now, suppose that a solution $r = r(\theta)$ satisfies an initial condition $r(\theta_0) = r_0$, and that r_0 is so small that $1 + d_1(\theta)r(\theta) + d_2(\theta)r(\theta)^2 > 0$. Then from (2.3), r is holomorphic in r_0 and has a power series representation

$$r = \sum_{k=0}^{\infty} u_k(\theta) r_0^k. \quad (2.4)$$

The coefficients $u_k(\theta)$ of the right hand side are determined to be a 2π periodic functions of θ as follows:

If $\theta = \theta_0$, then we get

$$r_0 = \sum_{k=0}^{\infty} u_k(\theta_0) r_0^k$$

and

$$u_0(\theta_0) = 0, \quad u_1(\theta_0) = 1, \quad u_k(\theta_0) = 0 \text{ if } k \geq 2. \quad (2.5)$$

Since the solution r of (2.3) is identically zero if $r_0 = 0$, we have

$$u_0(\theta) \equiv 0.$$

Substitute (2.4) into (2.3). Then we obtain

$$\begin{aligned} u'_k(\theta) = & c_2(\theta) \sum_{k_1+k_2=k} u_{k_1}(\theta) u_{k_2}(\theta) \\ & + c_3(\theta) \sum_{k_1+k_2+k_3=k} u_{k_1}(\theta) u_{k_2}(\theta) u_{k_3}(\theta) - d_1(\theta) \sum_{k_1+k_2=k} u_{k_1}(\theta) u'_{k_2}(\theta) \\ & - d_2(\theta) \sum_{k_3+k_4=k} \sum_{k_1+k_2=k_3} u_{k_1}(\theta) u_{k_2}(\theta) u'_{k_4}(\theta). \end{aligned}$$

As $u_0(\theta) \equiv 0$, the right hand side contains only $u_1(\theta), u_2(\theta), \dots, u_{k-1}(\theta)$. Therefore $u_k(\theta)$ is determined from integrating the right hand side and seeing (2.5). In this way, we get

$$u_1(\theta) \equiv 1.$$

Moreover, since we have already known that (2.4) represents a periodic orbit, we determine $u_k(\theta)$ to be 2π periodic.

Now, recall

$$\begin{aligned} \begin{pmatrix} \eta \\ \zeta \end{pmatrix} &= \begin{pmatrix} 2 & 2 \\ \sqrt{\alpha} & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} 2 & 2 \\ \sqrt{\alpha} & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \begin{pmatrix} 2r(\cos \theta + \sin \theta) \\ \sqrt{\alpha}r(\cos \theta - \sin \theta) \end{pmatrix}. \end{aligned}$$

Then since $\eta = y - 1$, $\zeta = z$, we get

$$y - 1 = 2r(\cos \theta + \sin \theta), \quad z = \sqrt{\alpha}r(\cos \theta - \sin \theta).$$

From the first equation, we have

$$ty' = 2 \left\{ \frac{dr}{d\theta}(\cos \theta + \sin \theta) + r(\cos \theta - \sin \theta) \right\} t \frac{d\theta}{dt}$$

and from $z = ty'$ and the second equation,

$$\sqrt{\alpha}r(\cos \theta - \sin \theta) = 2 \left\{ \frac{dr}{d\theta}(\cos \theta + \sin \theta) + r(\cos \theta - \sin \theta) \right\} t \frac{d\theta}{dt}. \quad (2.6)$$

On the other hand, we obtain

$$c_2(\theta) = (\cos \theta - \sin \theta)g_2(\theta), \quad c_3(\theta) = (\cos \theta - \sin \theta)g_3(\theta)$$

where

$$g_2(\theta) = -\frac{1}{\alpha} - \frac{2\alpha - 1}{\alpha} \sin 2\theta, \quad g_3(\theta) = -2(1 + \sin 2\theta)(\cos \theta + \sin \theta).$$

Hence from (2.3) we get

$$\frac{dr}{d\theta} = (\cos \theta - \sin \theta) \frac{g_2(\theta)r^2 + g_3(\theta)r^3}{1 + d_1(\theta)r + d_2(\theta)r^2}$$

and from (2.6)

$$\sqrt{\alpha} = 2 \left\{ \frac{g_2(\theta)r + g_3(\theta)r^2}{1 + d_1(\theta)r + d_2(\theta)r^2}(\cos \theta + \sin \theta) + 1 \right\} t \frac{d\theta}{dt}. \quad (2.7)$$

Indeed, from (2.2) and $1 + d_1(\theta)r + d_2(\theta)r^2 \neq 0$ we have $d\theta/d\tau \neq 0$. Hence $\cos \theta - \sin \theta \neq 0$ holds for almost every τ . Since the solution $x(t)$ of (E) is two times continuously differentiable, (r, θ) is continuously differentiable in t if $r > 0$. Therefore for almost every t we get $dt/d\tau \neq 0$ from $d\theta/d\tau \neq 0$ and $d\tau/dt = (d\tau/d\theta)(d\theta/dt)$, and so $\cos \theta - \sin \theta \neq 0$. Hence (2.7) holds for almost every t , and for all t from the continuous differentiability of (r, θ) . Thus we have

$$\left\{ \frac{g_2(\theta)r + g_3(\theta)r^2}{1 + d_1(\theta)r + d_2(\theta)r^2}(\cos \theta + \sin \theta) + 1 \right\} \frac{d\theta}{dt} = \frac{\sqrt{\alpha}}{2} \frac{1}{t}$$

and from $r = \sum_{k=0}^{\infty} u_k(\theta)r_0^k$ and $1 + d_1(\theta)r + d_2(\theta)r^2 \neq 0$,

$$\left(1 + \sum_{k=1}^{\infty} w_k(\theta)r_0^k \right) \frac{d\theta}{dt} = \frac{\sqrt{\alpha}}{2} \frac{1}{t}$$

where $w_k(\theta)$ are 2π periodic functions. If $\theta = \theta_0$ as $t = t_0$, then we integrate this from t_0 to t and obtain

$$\theta - \theta_0 + \sum_{k=1}^{\infty} \int_{\theta_0}^{\theta} w_k(\theta) d\theta r_0^k = \frac{\sqrt{\alpha}}{2} \log \frac{t}{t_0}.$$

Here, let a_k be the means of $w_k(\theta)$ and $b_k(\theta) = w_k(\theta) - a_k$. Then we get

$$L(r_0)(\theta - \theta_0) + \sum_{k=1}^{\infty} \int_{\theta_0}^{\theta} b_k(\theta) d\theta r_0^k = \frac{\sqrt{\alpha}}{2} \log \frac{t}{t_0}$$

where

$$L(r_0) = 1 + \sum_{k=1}^{\infty} a_k r_0^k.$$

If $L(r_0) = 0$, then we obtain a contradiction that the left hand side is bounded and the right hand side is unbounded. This implies $L(r_0) \neq 0$, and

$$L(r_0) > 0,$$

for $L(r_0)$ is holomorphic in r_0 and $L(0) = 1 > 0$. Hence if we put

$$P(\theta, r_0) = \frac{1}{L(r_0)} \sum_{k=1}^{\infty} b_k(\theta) r_0^k, \quad K(r_0) = \frac{\sqrt{\alpha}}{2L(r_0)},$$

then we have

$$\theta - \theta_0 + \int_{\theta_0}^{\theta} P(\theta, r_0) d\theta = K(r_0) \log \frac{t}{t_0} \quad (2.8)$$

where $K(r_0)$ is determined to be a positive constant depending on r_0 holomorphically. Moreover let $Q(\theta, r_0)$ denote an indefinite integral of $P(\theta, r_0)$ with mean 0. Then $Q(\theta, r_0)$ is determined to be a continuous function of (θ, r_0) 2π periodic and differentiable in θ and holomorphic in r_0 such that $Q(\theta, 0) = 0$, and we obtain (1.3).

In addition, from (T) and the transformations which changes (E) into (2.2) we obtain a solution of (E) as

$$\begin{aligned} x &= \psi(t)y^{1/\alpha} = \psi(t) \left\{ 1 + 2(r_0 + \sum_{k=2}^{\infty} u_k(\theta)r_0^k)(\cos \theta + \sin \theta) \right\}^{1/\alpha} \\ &= 4^{-1/\alpha} t^{1/2} \left(1 + \sum_{k=1}^{\infty} x_k(\theta)r_0^k \right) \end{aligned}$$

where $x_k(\theta)$ are 2π periodic function. Thus we get (1.2).

Now we show that θ is an increasing function of t . Since the periodic orbits of (S) do not pass a critical point, we might suppose that for some sequence $\{s_n\}_{n=1}^{\infty}$, we have $z \neq 0$ if $s \neq s_n$. Then from $dy/ds = 4\alpha yz$ of (S) and $z = ty'$, we get

$$\frac{ds}{dy} = \frac{1}{4\alpha yz}, \quad \frac{dt}{dy} = \frac{t}{z}.$$

Hence t, s are differentiable function of y and satisfy

$$\frac{ds}{dy} = \frac{1}{4\alpha y t} \frac{dt}{dy}$$

for $s \neq s_n$. Integrating both sides on an interval of y where $s \neq s_n$ holds, we get

$$s = \int \frac{1}{4\alpha y t} dt + \text{const.}$$

and

$$\frac{ds}{dt} = \frac{1}{4\alpha y t}.$$

Therefore from $\tau = 2\alpha^{3/2}s$ we have

$$\frac{dt}{d\tau} = \frac{2yt}{\alpha^{1/2}} > 0$$

for $\tau \neq 2\alpha^{3/2}s_n$. This implies that t is a function of τ , and for all t we obtain

$$\frac{d\theta}{dt} = \frac{d\theta}{d\tau} \frac{d\tau}{dt} > 0, \quad (2.9)$$

since $1 + d_1(\theta)r(\theta) + d_2(\theta)r(\theta)^2 > 0$ and $d\theta/dt \neq 0, \infty$ from (2.7). That is, θ is an increasing function of t .

Finally we show that the left hand side of (1.3) is an increasing function of θ . For this, we differentiate both sides of (2.8) and get

$$(1 + P(\theta, r_0)) \frac{d\theta}{dt} = \frac{K(r_0)}{t}.$$

This implies

$$1 + P(\theta, r_0) > 0$$

from (2.9), and that the left hand side of (2.8), namely that of (1.3) is an increasing function of θ .

Here for completing the proof of Theorem 1, it just remains to show that its conclusion follows for all $(y_0, z_0) \in \Delta$.

3. On nonperiodic orbits of (S)

Let us consider the critical point $(0, 0)$. From (S) we get

$$\frac{d}{ds}(z - \sigma y) = -\alpha^2 y^2 \left(y - \frac{\alpha^2 - 4\sigma^2}{\alpha^2} \right)$$

on a line $z = \sigma y$ (σ : a constant). Using this and discussing as in the proof of Lemma 1 of [13], we conclude that if an orbit (y, z) of (S) tends to $(0, 0)$, then this satisfies

$$\lim_{y \rightarrow 0} \frac{z}{y} = \pm \frac{\alpha}{2}, \quad \pm \infty.$$

If the orbit (y, z) of (S) satisfies $\lim_{y \rightarrow 0} z/y = \alpha/2$, then from the proof of Lemma 3.2 of [14] we conclude the uniqueness of such an orbit and represent this orbit as

$$z = \frac{\alpha}{2}y(1 + \sum_{n=1}^{\infty} z_n y^n) \quad (z_n : \text{constants})$$

in the neighbourhood of $y = 0$. Moreover, from this we have a solution of (E) represented as (1.4) in the neighbourhood of $t = 0$. In the same way, we conclude the uniqueness of the orbit (y, z) satisfying $\lim_{y \rightarrow 0} z/y = -\alpha/2$, and express this as

$$z = -\frac{\alpha}{2}y(1 + \sum_{n=1}^{\infty} \tilde{z}_n y^n) \quad (\tilde{z}_n : \text{constants})$$

in the neighbourhood of $y = 0$. From this, we obtain a solution of (E) represented as (1.5) in the neighbourhood of $t = \infty$.

Now, notice that a solution $z(y)$ of (R) represents an orbit of (S). Then if we discuss as in Lemma 7 of [17] or in Lemma 10 of [18], we conclude the following:

LEMMA. *The orbit $z(y)$ is bounded at every y ($0 < y < \infty$) in the closure of its domain, and is not continuable to $y = \infty$.*

This implies that if (y, z) is the unique orbit with $\lim_{y \rightarrow 0} z/y = \alpha/2$, then in the region $z > 0$, y increases as s does, since $dy/ds = 4\alpha yz$, and (y, z) enters the region $z < 0$. As stated above, the orbits of (2.1) are symmetric with respect to the axis of abscissas, and so are those of (S). Therefore (y, z) satisfies $\lim_{y \rightarrow 0} z/y = -\alpha/2$, and the unique orbits satisfying $\lim_{y \rightarrow 0} z/y = \alpha/2$ and $\lim_{y \rightarrow 0} z/y = -\alpha/2$ are the same. So we denote such an orbit as Γ .

Lemma implies again that if (y, z) is an orbit of (S) lying outside $\Delta \cup \Gamma$, then y is continuable to $y = 0$ and (y, z) must satisfy

$$\lim_{y \rightarrow 0} \frac{z}{y} = \pm\infty.$$

So, put $z = yw^{-1}$ and discuss as in Section 5 of [11]. Then we get

$$z^{-1} = Cy^{1/\alpha-1} \left\{ 1 + \sum_{m+n>0} w_{mn} y^m (Cy^{1/\alpha})^n \right\} \quad (3.1)$$

where C, w_{mn} are constants, and a solution of (E) represented as (1.6), (1.7) respectively if $z > 0, z < 0$. (3.1) implies that in the undrawn parts of Figure, the orbit (y, z) behaves as

$$\lim_{y \rightarrow 0} z = \pm\infty, c, 0 \quad (c : \text{a nonzero constant})$$

respectively, if $0 < \alpha < 1, \alpha = 1, \alpha > 1$. From the above discussion on Γ , (y, z) surrounds Γ as drawn in Figure.

Now, let $x(t)$ be a solution of (E), (I) and define the orbit (y, z) of (S) from applying (T) to $x(t)$. Then as in Lemma 2 of [13], we conclude that if $\omega_- < t < \omega_+$ is the domain of $x(t)$, then (y, z) does not tend to a regular point of (S) as $t \rightarrow \omega_{\pm}$. Therefore if $(y_0, z_0) \in \Gamma$, then the orbit (y, z) passing (y_0, z_0) moves all over Γ as t varies from ω_- to ω_+ . Moreover from the above discussion, we get $\omega_- = 0, \omega_+ = \infty$, and (1.4), (1.5) in the

neighbourhoods of $t = 0$, $t = \infty$, respectively. This shows Theorem 2. In the same way, if (y_0, z_0) lies outside $\Delta \cup \Gamma$, then the orbit (y, z) passing (y_0, z_0) satisfies (3.1), from which we get $0 < \omega_- < \omega_+ < \infty$, and (1.6), (1.7). This proves Theorem 3.

4. On the region of the existence of the periodic orbits

In Section 2, we transformed (E) into (2.2), namely

$$\frac{dr}{d\tau} = c_2(\theta)r^2 + c_3(\theta)r^3, \quad \frac{d\theta}{d\tau} = 1 + d_1(\theta)r + d_2(\theta)r^2 \quad (4.1)$$

in terms of the transformations (T), $(y, z) = (1 + \eta, \zeta)$,

$$\begin{pmatrix} \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ \sqrt{\alpha} & -\sqrt{\alpha} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix},$$

$(u, v) = (r \cos \theta, r \sin \theta)$, and $\tau = 2\alpha^{3/2}s$. Under the assumption

$$1 + d_1(\theta)r + d_2(\theta)r^2 > 0 \quad (4.2)$$

we got the analytical expression (1.2) of the solution $x(t)$ of (E). Here, suppose that $x(t)$ is not the particular solution $\psi(t)$, and let $(r(\tau), \theta(\tau))$ denote a solution of (4.1) got from applying those transformations to $x(t)$. Then we show that (4.2) holds for $r = r(\tau)$, $\theta = \theta(\tau)$.

For this, suppose the contrary. Then if

$$\Omega = \{(r(\tau), \theta(\tau)) : \tau \in \mathbf{R}\},$$

there exists a point $(r_0, \theta_0) \in \Omega$ such that

$$1 + d_1(\theta_0)r_0 + d_2(\theta_0)r_0^2 = 0.$$

In this case, if

$$c_2(\theta_0)r_0^2 + c_3(\theta_0)r_0^3 = 0,$$

then (r_0, θ_0) is a critical point of (4.1) and $(r(\tau), \theta(\tau))$ is constant in τ . Hence so are (u, v) , (η, ζ) , and (y, z) . That is, (y, z) is a critical point of (E), which is contrary to the assumption that $x(t)$ is not the particular solution $\psi(t)$. Therefore we have

$$c_2(\theta_0)r_0^2 + c_3(\theta_0)r_0^3 \neq 0.$$

Now, assume that there exists a sequence $\{(r_n, \theta_n)\}$ such that $(r_n, \theta_n) \neq (r_0, \theta_0)$, $(r_n, \theta_n) \rightarrow (r_0, \theta_0)$ as $n \rightarrow \infty$, and

$$1 + d_1(\theta_n)r_n + d_2(\theta_n)r_n^2 = 0.$$

Then since $1 + d_1(\theta)r + d_2(\theta)r^2$ is a holomorphic function of (r, θ) , we get

$$1 + d_1(\theta)r + d_2(\theta)r^2 \equiv 0$$

in terms of the theorem of invariance of analytic relations. Hence from (4.1) we have

$$\theta \equiv \theta_0$$

for all τ , and from the above transformations

$$\tan \theta_0 = \frac{\sqrt{\alpha}(y-1) - 2z}{\sqrt{\alpha}(y-1) + 2z}.$$

Therefore we obtain

$$z = k(y-1), \quad k = \frac{\sqrt{\alpha}(1 - \tan \theta_0)}{2(1 + \tan \theta_0)}. \quad (4.3)$$

Here, let τ tend to $\pm\infty$. Then $s \rightarrow \pm\infty$ and from Poincaré-Bendixon's theorem, the orbit (y, z) of (S) tends to the critical point $(1, 0)$ along the line (4.3) as $s \rightarrow \infty$ or $s \rightarrow -\infty$. This is absurd, for $(1, 0)$ is a centre. Therefore there does not exist such $\{(r_n, \theta_n)\}$ and there exists a neighbourhood U_1 of (r_0, θ_0) such that

$$1 + d_1(\theta)r + d_2(\theta)r^2 \neq 0$$

for $(r, \theta) \in U_1 - \{(r_0, \theta_0)\}$.

Thus for the solution (r, θ) of (4.1) it follows from its second equation that τ is a function of θ in $U_1 - \{(r_0, \theta_0)\}$. Hence r is a function of θ in the same region and in U_1 particularly for $(r, \theta) \in \Omega$, since $(r_0, \theta_0) \in \Omega$. Therefore, put $r = r(\theta)$. Then r satisfies $r(\theta_0) = r_0$ and (2.3), namely

$$\frac{dr}{d\theta} = \frac{c_2(\theta)r^2 + c_3(\theta)r^3}{1 + d_1(\theta)r + d_2(\theta)r^2}.$$

From this, we obtained

$$\left\{ \frac{g_2(\theta)r + g_3(\theta)r^2}{1 + d_1(\theta)r + d_2(\theta)r^2} (\cos \theta + \sin \theta) + 1 \right\} \frac{d\theta}{dt} = \frac{\sqrt{\alpha}}{2} \frac{1}{t} \quad (4.4)$$

in Section 2. This implies $d\theta/dt \neq 0$.

Furthermore, suppose that $(r(\theta), \theta) \in U_1 - \{(r_0, \theta_0)\}$ on a closed interval $[\theta_1, \theta]$, and that $t = t_1$ if $\theta = \theta_1$. Then, integrating both sides of (4.4) from θ_1 to θ , we get

$$\begin{aligned} & \int_{\theta_1}^{\theta} \frac{g_2(\theta)r(\theta) + g_3(\theta)r(\theta)^2}{1 + d_1(\theta)r(\theta) + d_2(\theta)r(\theta)^2} (\cos \theta + \sin \theta) d\theta \\ & + \theta - \theta_1 = \frac{\sqrt{\alpha}}{2} \log \frac{t}{t_1}. \end{aligned} \quad (4.5)$$

Here we show

$$(g_2(\theta_0)r(\theta_0) + g_3(\theta_0)r(\theta_0)^2)(\cos \theta_0 + \sin \theta_0) \neq 0. \quad (4.6)$$

In fact, if $g_2(\theta_0)r(\theta_0) + g_3(\theta_0)r(\theta_0)^2 = 0$, then we have

$$\begin{aligned} c_2(\theta_0)r(\theta_0)^2 + c_3(\theta_0)r(\theta_0)^3 &= (\cos \theta_0 - \sin \theta_0) \\ &\times (g_2(\theta_0)r(\theta_0) + g_3(\theta_0)r(\theta_0)^2)r(\theta_0) = 0. \end{aligned}$$

Hence $(r(\theta_0), \theta_0)$ is a critical point of (4.1) and from the above discussion we obtain the contradiction that (y, z) is the critical point. Therefore we obtain $g_2(\theta_0)r(\theta_0) + g_3(\theta_0)r(\theta_0)^2 \neq 0$.

Next, if $\cos \theta_0 + \sin \theta_0 = 0$, then we get

$$\sin 2\theta_0 = -1,$$

and

$$d_1(\theta_0) = \frac{1}{\alpha}(\cos \theta_0 + \sin \theta_0)\{2\alpha + 1 + (2\alpha - 1) \sin 2\theta_0\} = 0$$

$$d_2(\theta_0) = 2(\sin 2\theta_0 + 1)^2 = 0.$$

Since $r(\theta_0) = r_0$, this implies a contradiction

$$1 + d_1(\theta_0)r(\theta_0) + d_2(\theta_0)r(\theta_0)^2 = 1.$$

Therefore we have $\cos \theta_0 + \sin \theta_0 \neq 0$ and (4.6).

From (4.6), we obtain

$$\int_{\theta_1}^{\theta} \frac{g_2(\theta)r(\theta) + g_3(\theta)r(\theta)^2}{1 + d_1(\theta)r(\theta) + d_2(\theta)r(\theta)^2} (\cos \theta + \sin \theta) d\theta \rightarrow \pm\infty$$

as $\theta \rightarrow \theta_0$, and then from (4.5)

$$t \rightarrow \infty, 0.$$

On the other hand, take (t_0, A, B) of (I) such that the orbit (r, θ) of (4.1) passes (r_0, θ_0) . Then for this (r, θ) , we obtain (4.5). Since $x(t) \rightarrow A$, $x'(t) \rightarrow B$ as $t \rightarrow t_0$ and (r, θ) is continuous in $x(t)$, $x'(t)$ from the above transformations, we get $\theta \rightarrow \theta_0$ as $t \rightarrow t_0$. Therefore from the above discussion we conclude $t \rightarrow \infty, 0$ as $\theta \rightarrow \theta_0$, which implies a contradiction

$$t_0 = \infty, 0$$

Hence we have (4.2).

Now we prove Theorem 1 as follows: From Poincaré-Bendixon's theorem, the unique existence of Γ , and the symmetricity of the orbits of (S), the orbits contained in Δ are all periodic ones surrounding the critical point $(1, 0)$. In addition, (4.2) holds. Therefore the discussion done in Section 2 is valid and if $(y_0, z_0) \in \Delta$ and $(y_0, z_0) \neq (1, 0)$, then we conclude that the solution of (E), (I) exists for $0 < t < \infty$ and is represented as (1.2). This completes the proof of Theorem 1.

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